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GENERAL THEOREMS ON ENERGY TRANSPORT BY HOMOGENEOUS WAVES†

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Rigorous proofs of the equivalence of the kinematic and energy definitions of the group velocity of the fundamental modes of propagating homogeneous waves in a Lagrangian continuum system of general form are given. Plane, cylindrical and spherical waves in an unbounded medium are considered, together with waves in waveguides, and plane and cylindrical waves in layers. The applicability of the results obtained to problems in the theory of elasticity is indicated. Formulae are given for the energies of elastic waves.

EXISTING general proofs of the identity of the kinematic and energy definitions, based on variational approaches for Lagrangian continuum systems [1–5], are not mathematically impeccable and usually refer only to plane waves.

1. STATEMENT OF THE PROBLEM

Consider a continuum system of general form in three-dimensional Euclidean space. Let the state of the system be described by a Lagrangian density L, which does not depend explicitly on the coordinates x_1, x_2, x_3 and time t, and is a homogeneous quadratic form of the characteristic parameter-functions $u_i = u_i(x_1, x_2, x_3, t) = u_i(x_j, t)$ and their first derivatives $u_{i,t} = \partial u_i / \partial t$, $u_{ij} = \partial u_i / \partial x_j$. We thus have

$$L = L(u_{i,t}, u_{i,j}, u_i) \tag{1.1}$$

From Hamilton's principle for this system we obtain the Euler–Lagrange equations

$$L_{it,t} + L_{ij,j} - L_i = 0, \quad i = 1,2,\ldots,n \tag{1.2}$$

Here and henceforth repeated subscripts imply summation and the following notation is used

$$L_{it,t} = \partial L_{it} / \partial t, \quad L_{ij,j} = \partial L_{ij} / \partial x_j$$

$$L_{it} = \partial L / \partial u_{i,t}, \quad L_{ij} = \partial L / \partial u_{i,j}, \quad L_i = \partial L / \partial u_i$$

Multiplying (1.2) by u_{it} and performing standard transformations, we obtain the equation of continuity for the energy E

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$$\partial E / \partial t + \nabla \cdot \mathbf{J} = 0 \tag{1.3}$$

$$E = u_{i,t} L_{it} - L, \quad J_j = u_{i,t} L_{ij} \tag{1.4}$$

where $\mathbf{J}$ is the specific energy flux vector (the Poynting vector) defining the rate and direction of energy transport.

The physical meaning of the vector $\mathbf{J}$ follows from the integral form of Eq. (1.3) over the volume V (Poynting's theorem)

$$\frac{\partial}{\partial t} \int_V E d\upsilon + \int_{\partial V} \mathbf{n} \cdot \mathbf{J} ds = 0 \tag{1.5}$$

where $\mathbf{n}$ is the unit outward normal to the boundary ∂V of the volume.

We shall assume that the Lagrangian system under consideration is in a free steady oscillatory mode with frequency ω. Then u_i, L, all their partial derivatives, and consequently, also E and $\mathbf{J}$, will be periodic in t with period $T = 2\pi/\omega$. Hence averaging (1.5) over the period, we obtain the equality

$$\int_{\partial V} \mathbf{n} \cdot \langle \mathbf{J} \rangle ds = 0$$

$$\langle \ldots \rangle = \langle \ldots \rangle_\omega = \frac{1}{T} \int_0^T (\ldots) dt, \quad T = \frac{2\pi}{\omega} \tag{1.6}$$

expressing the energy balance in the volume V for periodic processes.

Below we shall only consider homogeneous normal propagating waves (HWs), which satisfy the homogeneous equations of motion (1.2), and for media with boundaries, the homogeneous boundary conditions also. The set of these conditions generates a dispersion relation

$$D(\omega, \alpha_j) = 0 \tag{1.7}$$

connecting the frequency ω ($\omega > 0$) and the wave numbers α_j. For an HW $\operatorname{Im}\alpha_j = 0$ for all j.

The set of all real solutions α_j to Eq. (1.7) for given ω will be called the polar set.

The group velocity vector $\mathbf{c}_g$

$$c_{gj} = \partial \omega / \partial \alpha_j = -D_{,\alpha_j} / D_{,\omega} \tag{1.8}$$

is a kinematic characteristic of the HW and is assumed to be non-zero.

The main problem is to establish, by a rigorous proof, that for the fundamental types of HW the group velocity is the energy transport velocity, i.e.

$$\mathbf{c}_g = \mathbf{c}_e \tag{1.9}$$

where, depending on the form of HW

$$\mathbf{c}_e = \langle \mathbf{J} \rangle / \langle E \rangle \tag{1.10}$$

or

$$\mathbf{c}_e = \int_S \langle \mathbf{J} \rangle ds \Big/ \int_S \langle E \rangle ds, \quad S \subset \partial V \tag{1.11}$$

if S has a unique normal at every point.

From (1.3)–(1.6) $\mathbf{c}_e$ is the rate of the mean energy transport across unit surface area with normal $\mathbf{n}$ in case (1.10) or across the surface S in case (1.11).

2. PLANE HWS IN AN UNBOUNDED MEDIUM

We shall assume that the plane HW of the form

$$u_i = f_i(\eta), \quad \eta = \omega t - \alpha_j x_j \tag{2.1}$$

satisfies the equations of motion (1.2), and the functions f_i are periodic in t with period T.

Lemma 2.1. The plane HW (2.1) satisfies the condition $\delta\langle L\rangle = 0$ for all variations δu_i which preserve the dependence on η and possess periodicity with the same period as the wave itself.

Proof. We have

$$\begin{gathered}\delta\langle L\rangle = \delta\left\langle L(u_{i,t}, u_{i,j}, u_i)\right\rangle = \left\langle L_{it}\delta u_{i,t} + L_{ij}\delta u_{i,j} + L_i\delta u_i\right\rangle = \\ = \left\langle(-L_{it,t} - L_{ij,j} - L_i)\delta u_i\right\rangle + \left\langle\Phi_{,t}\right\rangle + \left\langle\Phi_{j,j}\right\rangle \\ (\Phi = \Phi(\eta) = L_{it}\delta u_i, \quad \Phi_j = \Phi_j(\eta) = L_{ij}\delta u_i)\end{gathered} \tag{2.2}$$

The first term on the right-hand side of (2.2) vanishes because of the equations of motion (1.2), and the second because of the periodicity of $\Phi(\eta)$ in t with period T. Finally, the final integral is in fact equal to zero for the same reason as the second, because the derivative with respect to x_j can be expressed in terms of the derivative with respect to t

$$\Phi_{j,k} = -\alpha_k\Phi'_j, \quad \Phi_{j,t} = \omega\Phi'_j, \quad \Phi_{j,k} = -\alpha_k\omega^{-1}\Phi_{j,t} \tag{2.3}$$

(the prime denotes the derivative with respect to η).

We emphasize that the conditions of Lemma 2.1 do not require the waves $u_i + \delta u_i$ to be solutions of Eqs (1.2).

Lemma 2.2. For any plane HW (2.1) $\langle L\rangle = 0$.

Proof. Following [5] we replace u_i by $(1+\varepsilon)u_j$, $0 < \varepsilon \ll 1$. In this case the variation $\delta u_i = \varepsilon u_i$ will satisfy the conditions of Lemma 2.1, and hence $\delta\langle L\rangle = 0$. But because L is a homogeneous quadratic function of its arguments, $0 = \delta\langle L\rangle = \varepsilon^2\langle L\rangle$, and so $\langle L\rangle = 0$.

To formulate the following lemma we represent the Lagrangian L in the form

$$L = L(\omega, \alpha_j, f_i(\eta)) \tag{2.4}$$

separating the explicit dependence of L on ω and α_j from the dependence of L on ω and α_j through η.

Lemma 2.3. If ω and α_j are respectively the frequency and wave numbers of the HW (2.1), then for any Ω and β_j

$$\left\langle L(\omega, \alpha_j, f_i(\xi))\right\rangle_\Omega = 0, \quad \xi = \Omega t - \beta_j x_j$$

Since L is a homogeneous quadratic form in $f_i(\eta)$ and $f_i'(\eta)$, and $\langle f_i^{(\cdot)}(\eta) f_k^{(\cdot)}(\eta)\rangle_\omega = \langle f_i^{(\cdot)}(\xi) f_k^{(\cdot)}(\xi)\rangle_\Omega$, Lemma 2.3 is obvious.

Theorem 2.1. Formulae (1.8)–(1.10) hold for the plane HW (2.1).

Proof. Throughout the proof the equals sign denotes the equality of terms to first order of smallness inclusive.

Consider a plane HW, similar to the HW (2.1), of the form

$$u_i + \delta u_i = f_i(\eta + \delta\eta) + \delta f_i(\eta + \delta\eta), \quad \delta\eta = \delta\omega t - \delta\alpha_j x_j \tag{2.5}$$

for which the variation $\delta f_i(\eta)$ is periodic with the same periodicity as $f_i(\eta)$.

The frequency $\omega+\delta\omega$ and wave numbers $\alpha_j+\delta\alpha_j$ of HW (2.5) related by the dispersion equation (1.7), and consequently

$$\delta\omega / \delta\alpha_j = c_{gj} \tag{2.6}$$

Note that the HW (2.5) already has a changed period, equal to $T+\delta T = 2\pi/\Omega$; $\Omega=\omega+\delta\omega$.

Apart from linear terms we can write

$$\begin{aligned} &\delta f_i(\eta+\delta\eta) = \delta f_i(\eta) \equiv \delta_c f_i(\eta) \\ &f_i(\eta+\delta\eta) = f_i(\eta) + \delta_\eta f_i(\eta), \quad \delta_\eta f_i = f_i'(\eta)\delta\eta \end{aligned} \tag{2.7}$$

which enables the variation δu_i from (2.5) to be represented in the form

$$\delta u_i = \delta_c f_i + \delta_\eta f_i \tag{2.8}$$

($\delta_\eta f_i$ is the variation caused only by varying η, while $\delta_c f_i$ is the variation in f_i when η is not affected). According to (2.7) $\delta_c f_i$ can to a first approximation be taken to be a function having both periods T and $T+\delta T$.

From Lemma 2.2 we have

$$\langle L(u_i+\delta u_i)\rangle_\Omega = 0 \tag{2.9}$$

On the other hand

$$\langle L(u_i+\delta u_i)\rangle_\Omega = \langle L(u_i)\rangle_\Omega + \langle \delta L\rangle_\Omega \tag{2.10}$$

Using the obvious formulae

$$u_{i,t} = \omega f_i', \quad u_{i,j} = -\alpha_j f_i' \tag{2.11}$$

and representation (1.1) for L, the last term in (2.10) can be transformed as follows:

$$\langle \delta L\rangle_\Omega = \langle \delta L(\omega f_i', -\alpha_j f_i', f_i)\rangle_\Omega = I_{\omega\alpha} + I_c + I_\eta \tag{2.12}$$

Here

$$\begin{aligned} &I_{\omega\alpha} = \langle L_{it} f_i'\rangle_\Omega \delta\omega - \langle L_{ij} f_i'\rangle_\Omega \delta\alpha_j \\ &I_\gamma = \langle L_{it}\omega\delta_\gamma f_i' - L_{ij}\alpha_j\delta_\gamma f_i' + L_i\delta_\gamma f_i\rangle_\Omega, \quad \gamma = c, \eta \end{aligned} \tag{2.13}$$

Note that for quantities of the first order of smallness

$$\langle \ldots\rangle_\Omega = \langle \ldots\rangle_\omega \tag{2.14}$$

From (2.14) and Lemma 2.1 $I_c = 0$, since only the variation $\delta_c f_i$ participates in I_c, preserving the dependence on η.

Further, in the notation of (2.4)

$$I_\eta = \langle L(\omega, \alpha_j, f_i(\eta+\delta\eta))\rangle_\Omega - \langle L(\omega, \alpha_j, f_i(\eta))\rangle_\Omega$$

and using Lemma 2.3

$$I_\eta = -\langle L(u_i)\rangle_\Omega$$

Using the values of I_c and I_η obtained, from (2.9), (2.10) and (2.12) we have

$$I_{\omega\alpha} = 0 \tag{2.15}$$

After multiplying (2.15) by ω and using (2.11), (2.13), (2.14) and Lemma 2.2 ($\langle L\rangle = 0$), we finally obtain

$$\delta\omega / \delta\alpha_j = \langle J_j\rangle / \langle E\rangle$$

which by (2.6) also proves Theorem 2.1.

This proof of Theorem 2.1 is closest in form to that given in [5]. However, Lemmas 2.1 and 2.2 were formulated in [5] for integrals over rectangular volumes of $\langle L\rangle$ with edge lengths equal to an integer number of wavelengths, and Theorem 2.1 was only proved for variations in ω and α_j that preserved the period T.

3. CYLINDRICAL HWS IN AN UNBOUNDED MEDIUM

Consider a Lagrangian system with no dependence on the variable x_3. A cylindrical HW satisfying the requirement that energy transport through a surface of radius $r = (x_1^2 + x_2^2)^{1/2}$ remains finite as $r \to \infty$, has the form

$$u_i = \int_\Gamma f_i(\omega, \alpha_m, \eta) d\gamma, \quad \eta = \omega t - \alpha_m x_m \tag{3.1}$$

where Γ is the connected part of the polar set, and the subscript m takes the values 1 and 2.

We shall assume that in a polar system of coordinates

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta \tag{3.2}$$

for the HW (3.1) the asymptotic representations

$$u_i = r^{-1/2} f_{ia}(\omega, \alpha_{mc}, \eta), \quad \eta = \omega t - r(\alpha_{1c}\cos\theta + \alpha_{2c}\sin\theta) \tag{3.3}$$

hold for large r, where the f_{ia} are also periodic in t with period T, and the α_{mc} are defined by the system of equations

$$D(\omega, \alpha_m) = 0, \quad c_{g\theta} = -c_{g1}\sin\theta + c_{g2}\cos\theta = 0 \tag{3.4}$$

In terms of the stationary-phase method [6], in the case of an oscillating integral (3.1) α_{1c} and α_{2c} are non-degenerate stationary points of the first order.

In (3.3) α_{1c} and α_{2c} are functions of θ. However, apart from terms of order $r^{-1/2}$

$$u_{i,m} = -\alpha_{mc} r^{-1/2} f'_{ia}, \quad f'_{ia} = \partial f_{ia} / \partial\eta \tag{3.5}$$

The identity of the structures of formulae (3.5) and (2.11) means that all the results of Sec. 2 hold for the cylindrical HWs (3.3) in the far field. Here α_j and f_i must be replaced by α_{mc} and $r^{-1/2} f_{ia}$.

Note that in a polar system of coordinates (3.2) the components of the Poynting vector

$$J_r = J_1\cos\theta + J_2\sin\theta, \quad J_\theta = -J_1\sin\theta + J_2\cos\theta$$

are given, according to (1.9), (1.10) and (3.4), by the relations

$$\langle J_r \rangle = c_{gr} \langle E \rangle, \quad \langle J_\theta \rangle = 0$$
$$(c_{gr} = c_{g1} \cos\theta + c_{g2} \sin\theta) \tag{3.6}$$

4. SPHERICAL HWS IN AN UNBOUNDED MEDIUM

Spherical HWs in a three-dimensional Lagrangian system are given in terms of plane HWs by the formula

$$u_i = \int_\Gamma f_i(\omega, \alpha_j, \eta) d\gamma, \quad \eta = \omega t - \alpha_j x_j \tag{4.1}$$

and unlike (3.1) Γ is now the connected component of the polar set in the three-dimensional space α_1, α_2, α_3.

Suppose that in a spherical system of coordinates

$$x_1 = R\cos\theta\sin\varphi, \quad x_2 = R\sin\theta\sin\varphi, \quad x_3 = R\cos\varphi$$

the spherical HW (4.1) has, for large R, a principal asymptotic term of the form

$$u_i = R^{-1} f_{ia}(\omega, \alpha_{jc}, \eta)$$
$$\eta = \omega t - R(\alpha_{1c} \cos\theta\sin\varphi + \alpha_{2c} \sin\theta\sin\varphi + \alpha_{3c} \cos\varphi) \tag{4.2}$$

with the values of α_{jc} given by the system of equations

$$D(\omega, \alpha_j) = 0, \quad c_{g\theta} = 0$$
$$c_{g\varphi} = c_{g1} \cos\theta\cos\varphi + c_{g2} \sin\theta\cos\varphi - c_{g3} \sin\varphi = 0 \tag{4.3}$$

As before, for an oscillating integral (4.1) system (4.3) defines non-degenerate stationary points by the stationary phase method.

Instead of (3.5) we now have, up to terms in R^{-1}

$$u_{i,j} = -\alpha_{jc} R^{-1} f'_{ia}$$

and it is obvious that Theorem 2.1 has been extended to the far field $R \gg 1$ and to spherical HWs (4.2).

For the far-field components of the vector $\mathbf{J}$ in a spherical system of coordinates, we have from (1.9), (1.10) and (4.2)

$$\langle J_R \rangle = c_{gR} \langle E \rangle, \quad \langle J_\theta \rangle = 0, \quad \langle J_\varphi \rangle = 0$$
$$(c_{gR} = c_{g1} \sin\varphi\cos\theta + c_{g2} \sin\varphi\sin\theta + c_{g3} \cos\varphi)$$

5. HWS IN WAVEGUIDES

Suppose that a waveguide is extended without limit along the x_1 axis and has a constant cross-section S. We will assume that an HW of the form

$$u_i = f_i(\omega, \alpha_1, x_2, x_3, \eta), \quad \eta = \omega t - \alpha_1 x_1 \tag{5.1}$$

satisfies the equations of motion (1.2) and homogeneous boundary conditions on the waveguide boundary $\partial S = \partial S_u \cup \partial S_\sigma$

$$u_i = 0, \quad \mathbf{x} \in \partial S_u \tag{5.2}$$

$$n_m L_{im} = 0, \quad \mathbf{x} \in \partial S_\sigma \tag{5.3}$$

where the subscript m takes the values 2 and 3.

For the HW (5.1) with the notation

$$\overline{(\ldots)} = \int_S (\ldots) ds \tag{5.4}$$

we have the following lemmas and theorem.

Lemma 5.1. The HW (5.1) satisfies the condition $\delta\langle \overline{L} \rangle = 0$ for all variations δu_i preserving the dependence on η, periodic in t with period T and satisfying the main boundary condition (5.2) $\delta u_i = 0$ on ∂S_u.

Lemma 5.2. For any HW (5.1) $\langle \overline{L} \rangle = 0$.

Lemma 5.3. If ω and α_1 are, respectively, the eigenfrequency and wave number of the HW (5.1), then for any Ω and β_1

$$\langle \overline{L}(\omega, \alpha_1, f_i(\Omega, \beta_1, x_2, x_3, \xi)) \rangle_\Omega = 0, \quad \xi = \Omega t - \beta_1 x_1$$

Theorem 5.1. Formulae (1.8)–(1.11) with $j = 1$ hold for the HW (5.1).

Lemmas 5.1–5.3 and Theorem 5.1 are proved by methods similar to those used for the corresponding lemmas and theorem in Sec. 2. We need only replace L by $\overline{L}$ as in (5.4) and make small changes having to do with integration over S and the different form of the HW. Thus, when proving Lemma 5.1, it is sufficient to take into account that formulae (2.3) for $\langle \overline{\Phi_{j,j}} \rangle$ only hold for $j = 1$, while for $j = 2, 3$ we have

$$\langle \overline{\Phi_{2,2}} \rangle + \langle \overline{\Phi_{3,3}} \rangle = \langle \overline{(L_{im} \delta u_i)_{,m}} \rangle = \left\langle \int_{\partial S} n_m L_{im} \delta u_i ds \right\rangle$$

which vanishes because of boundary conditions (5.2) and (5.3). In the proof of Theorem 5.1 it is necessary to include all the variations of f_i, ω and α which do not affect η in the variation $\delta_c f_i$ from (2.8). Finally, in formulae (2.9)–(2.16) one must place a bar over all quantities associated with L, put $j = 1$, and separately write out the derivatives with respect to $u_{i,2}$ and $u_{i,3}$.

Theorem 5.2. Theorem 5.1 also holds for systems in which the Lagrangian L satisfies all the conditions given in Sec. 1, but can also explicitly depend on x_2 and x_3, i.e.

$$L = L(x_2, x_3, u_{i,t}, u_{i,j}, u_i)$$

Note that in the proofs of Lemmas 2.1–2.3 and Theorem 2.1 the independence of the Lagrangian (1.1) from x_1, x_2 and x_3 was only necessary for formulae (2.3). However, these formulae are just for $j = 2$ and $j = 3$ and they are not required for the proof of Lemma 5.1.

Remark. In two-dimensional problems where there is no dependence on x_3, all the assertions remain true if S is taken to be the thickness of the waveguide. The results also hold for surface waves in the half-plane $x_2 \geqslant 0$, only now $S = [0, \infty)$.

6. PLANE HWS IN A LAYER

Suppose that in a layer extended without limit along x_1 and x_2 and of thickness S in x_3 there is a plane HW of the form

$$u_i = f_i(\omega, \alpha_1, \alpha_2, x_3, \eta), \quad \eta = \omega t - \alpha_1 x_1 - \alpha_2 x_2 \tag{6.1}$$

which satisfies the equations of motion (1.2) and the homogeneous boundary conditions (5.2) and (5.3) on the boundary ∂S, where $m=3$.

The integrals (5.4) are now one-dimensional. In this notation for the plane HW (6.1) there are analogues with Lemmas 5.1–5.3 and Theorems 5.1 and 5.2, in which $j=1$, 2; $m=3$; here L can depend explicitly on x_3. The changes in the proofs are obvious. We merely remark that in the formulation of Lemma 5.3 for the HW (6.1) the following quantity should appear

$$\langle \bar{L}(\omega, \alpha_1, \alpha_2, f_i(\Omega, \beta_1, \beta_2, x_3, \xi))\rangle_\Omega, \quad \xi = \Omega t - \beta_1 x_1 - \beta_2 x_2$$

which vanishes for all Ω, β_1 and β_2. Thus, for plane HWs (6.1) in a layer

$$c_{gj} = \langle \bar{J}_j \rangle / \langle \bar{E} \rangle, \quad j = 1,2$$

7. CYLINDRICAL HWS IN A LAYER

A cylindrical HW can be represented in terms of plane HWs in a layer by integrals of the form

$$u_i = \int_\Gamma f_i(\omega, \alpha_m, x_3, \eta)\, d\gamma, \quad \eta = \omega t - \alpha_m x_m \tag{7.1}$$

where the subscript m takes the values 1 and 2.

We will assume that in a cylindrical system of coordinates r, θ, x_3, where r and θ are defined by (3.2), for large r the HW (7.1) has an asymptotic representation of type (3.3), where f_{ia} also depends on x_3.

Repeating the arguments in Sec. 3, taking into account the changes associated with integration across the thickness of the layer, we have for the far field $r \gg 1$ that

$$\langle \bar{J}_r \rangle = c_{gr} \langle \bar{E} \rangle, \quad \langle \bar{J}_\theta \rangle = 0$$

This result, like the remark in Sec. 5, holds both for channel waves in a layer, and for surface waves in a half-space, where L can also depend explicitly on x_3.

8. ELASTIC HWS

The Lagrangian L and energy E for an anisotropic elastic medium have the form

$$L = T - W, \quad E = T + W \tag{8.1}$$

$$T = \tfrac{1}{2}\rho u_{i,t}^2, \quad W = \tfrac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl}, \quad \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{8.2}$$

For constant density ρ and elastic moduli c_{ijkl} the Lagrangian L satisfies the requirements of Sec. 1. Thus, for homogeneous anisotropic elastic media all the results of Secs 2–7 hold. For waveguides and layers, inhomogeneity of the medium over the cross-section is also allowed.

As has been proved before, for all the wave modes considered either $\langle L\rangle = 0$ or $\langle \bar{L}\rangle = 0$. Hence, using (8.1), for waves in an unbounded medium and for waves in a waveguide or layer we have, correspondingly

$$\langle T\rangle = \langle W\rangle, \quad \langle \bar{T}\rangle = \langle \bar{W}\rangle, \quad \langle E\rangle = 2\langle T\rangle, \quad \langle \bar{E}\rangle = 2\langle \bar{T}\rangle \tag{8.3}$$

For harmonic HWs

$$u_i = \begin{cases} \operatorname{Re} \upsilon_i \cos\eta - \operatorname{Im} \upsilon_i \sin\eta \\ \operatorname{Re} \upsilon_i \sin\eta + \operatorname{Im} \upsilon_i \cos\eta \end{cases}, \quad \eta = \omega t - \alpha_k x_k$$

where $k = 1$; $k = 1, 2$ or $k = 1, 2, 3$, and the amplitude functions υ_i can depend on ω, α_k and those spatial variables which do not occur in η, formulae (8.3), with (8.2) taken into account, can be given a form convenient for calculations (the asterisks denote complex conjugation)

$$\langle E\rangle = \tfrac{1}{2}\omega^2 \rho \upsilon_i \upsilon_i^*, \quad \langle \bar{E}\rangle = \tfrac{1}{2}\omega^2 \int_S \rho \upsilon_i \upsilon_i^* ds$$

Note that all the results obtained are of a local nature, and hence the requirements of an unbounded medium or an infinitely extended waveguide or layer are not stringent. It is only required that the remaining conditions be satisfied in the regions of the medium under investigation. For example, formulae (3.6) and (4.4) are also, respectively, valid for the far fields of cylindrical waves inside a half-plane and spherical waves inside a half-space, etc.

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